March 16, 2017

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Definition. A topological space $(X, \mathcal{T})$ is **compact** if every open cover has a finite subcover.

Logical statement. For each $\mathcal{C} \subset \mathcal{T}$ with $\bigcup \mathcal{C} = X$, there exists finite $\mathcal{F} \subset \mathcal{C}$ such that $\bigcup \mathcal{F} = X$.

Often, we need to talk about whether a subset A in a space X is compact.

Definition. A subset A in $(X, \mathcal{T})$ is **compact** if $(A, \mathcal{T}|_A)$ is compact.

Equivalently For each $\mathcal{C} \subset \mathcal{T}$ with $\bigcup \mathcal{C} \supset A$, there exists a finite $\mathcal{F} \subset \mathcal{C}$ such that $\bigcup \mathcal{F} \supset A$.

Related Concepts

Compact. Every open cover $\mathcal{C} \subset \mathcal{T}$ for X has a finite subcover $\mathcal{F} \subset \mathcal{C}$.

Sequentially Compact Every sequence in X has a subsequence converges in X.

Bolzano-Weierstrass Every infinite set in X has a cluster point in X.

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Review that $[a,b] \subset \mathbb{R}$ is compact

Let $\mathcal{C}$ be an open cover for $[a,b]$

Idea 1. Consider

$$L = \left\{ x \in [a,b] : \mathcal{C} \text{ has a finite subcover for } [a,x] \right\}$$

Since $a \in L$, we have $L \neq \emptyset$

Clearly, b is an upper bound of L

$\therefore$ $s = \sup L$ exists

It can be shown that $s < b$ leads to a contradiction

This method needs order, not even for $\mathbb{R}^n$

Idea 2. Assume $\mathcal{C}$ has no finite subcover

Subdivide $[a,b] = \left[a, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, b\right]$

Ask if $\mathcal{C}$ has finite subcover on each.

If true for both, then $\mathcal{C}$ has a finite subcover for $[a,b]$.

Therefore, at least one side needs infinite cover.

Repeat the subdivision argument on it.

At the end, $\exists$ nested intervals and a singleton. This leads to contradiction.

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Weakness

Completeness is used !!

However, compactness is not the same.

They are only related.

In the 2nd proof, order is avoided, but need to divide the space into two "halves".

Similar method obviously works for $\mathbb{R}^n$, $n \geq 2$.

In any case, a complete metric is needed; some notion of boundedness (called totally bounded) is enough.

So far, in the case of $\mathbb{R}^n$, within a bounded set or in the case of a complete metric set with suitably boundedness, a closed subset is compact.

Natural Question. How is closedness related to compactness?

下回分解